Meyer-Nieberg operators. Recent development.

ОБЪЕДИНЕННЫЙ ВОРКШОП ПО ФУНКЦИОНАЛЬНОМУ АНАЛИЗУ
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Introduction

We denote by X and Y Banach spaces, B_X the closed unit ball of X, by $\mathsf{L}(X,Y)$ the space of bounded linear operator from X to Y, E and F Banach lattices, $[x,y] := \{z \in E : x \leqslant z \leqslant y\}$ for $x, y \in E$,

$$\mathrm{sol}(A) := \bigcup_{a\in A} [-|a|, |a|]$$

the solid hull of $A \subseteq E$, and

$$E^a := \{x \in E : |x| \geqslant x_n \downarrow 0 \Rightarrow \|x_n\| \to 0\}$$

the order continuous part of E.

In what follows all operators are linear and bounded.

The theory of L-weakly compact (briefly, Lwc) sets and operators was created and developed in [Meyer-Nieberg: Math. Z. (1974)] in order to diversify the concept of weakly compact operators via imposing the Banach lattice structure on the range.

a) A subset A of F is an Lwc *set* if every disjoint sequence in sol(A) is norm-null.

b) An operator L(X, F) is an Lwc *operator* (briefly, $T \in$ Lwc(X, F)) if $T(B_X)$ is an Lwc subset of F.

Meyer-Nieberg proved that Lwc sets are relatively weakly compact (and hence Lwc operators are weakly compact). The next key fact goes back to [Meyer-Nieberg: Math. Z. (1974)]. It was precisely stated in [Burkinshaw and Dodds: Illinois J. Math. (1977)].

Proposition 1. *Let $A \subseteq E$ and $B \subseteq E'$ be nonempty bounded sets. Then every disjoint sequence in* sol(A) *is uniformly null on B iff every disjoint sequence in* sol(B) *is uniformly null on A.*

Since $\|f\| = \sup\{|f(x)| : x \in B_X\} = \sup\{|y(f)| : y \in B_{X''}\}$ then a sequence (f_n) in X' is uniformly null on B_X iff it is uniformly null on $B_{X''}$, and hence the next lemma follows from Proposition 1.

Lemma 2. *Let C be a nonempty bounded subset of F'. TFAE.*

$i)$ *C is an* Lwc *subset of F'.*

$ii)$ *Each disjoint sequence in B_F is uniformly null on C.*

$iii)$ *Each disjoint sequence in $B_{F''}$ is uniformly null on C.*

Meyer-Nieberg described the duals of Lwc operators via

Definition 3. *An operator* $T \in \mathsf{L}(E, Y)$ *is called an* Mwc *operator if* $\|Tx_n\| \to 0$ *for every disjoint bounded* (x_n) *in* E.

and proved in Satz.3 of [Meyer-Nieberg: Math. Z. (1974)]

$$S' \in \mathsf{Lwc}(Y', E') \Longleftrightarrow S \in \mathsf{Mwc}(E, Y).$$

$$T' \in \mathsf{Mwc}(F', X') \Longleftrightarrow T \in \mathsf{Lwc}(X, F).$$

In particular, the bi-duals of Lwc- (Mwc-operators) are Lwc (Mwc).

Replacing (in the definition of Lwc-operators) norm bounded sets by weakly compact sets and by order bounded sets, Bouras, Lhaimer and Moussa introduced in [BLM: Positivity (2018) and (2021)]

Definition 4. An operator $T \in \mathsf{L}(X,F)$ (resp. $T \in \mathsf{L}(E,F)$) is called a-Lwc (resp. o-Lwc) if T carries weakly compact subsets of X (resp. order bounded subsets of E) onto Lwc-subsets of F.

and described the duals of a-Lwc- and o-Lwc-operators via

Definition 5. An operator $T \in \mathsf{L}(E, Y)$ is called

$a)$ almost M-weakly compact (a-Mwc) if $f_n(Tx_n) \to 0$ for every w-convergent (f_n) in Y' and every disjoint bounded (x_n) in E.

An operator $T \in \mathsf{L}(E, F)$ is called

$b)$ order M-*weakly compact* (o-Mwc) if $f_n(Tx_n) \to 0$ for every order bounded (f_n) in F' and every disjoint bounded (x_n) in E.

Although the bi-duals of a-Lwc-/o-Lwc-operators need not to be a-Lwc/o-Lwc, there is the following semi-duality

Theorem 6. [BLM: Positivity (2018)]:

(i) $S' \in$ a-LW$(Y', E') \iff S \in$ a-MW(E, Y).

(ii) $T' \in$ a-MW$(F', X') \Rightarrow T \in$ a-LW(X, F).

And [BLM: Positivity (2021)]:

(iii) $S' \in$ o-LW$(F', E') \iff S \in$ o-MW(E, F).

(iv) $T' \in$ o-MW$(F', E') \Rightarrow T \in$ o-LW(E, F).

Definition 7. A bounded subset A of X is called:

a) a *Dunford–Pettis set* (or A is DP) if (f_n) is uniformly null on A for each w-null (f_n) in X' [Andrews: Math. Ann. (1979)].

b) a *limited set* if (f_n) is uniformly null on A for each w*-null (f_n) in X' [Bourgain and Diestel: Math. Nachr. (1984)].

In reflexive spaces, DP sets and limited sets agree with relatively compact sets. In general,

$$A \text{ is relatively compact } \Rightarrow A \text{ is limited } \Rightarrow A \text{ is DP.}$$

The unit ball B_X is not limited in X unless $\dim(X) < \infty$ (this fact was a solution of a long-standing open problem that each w*-null sequence in a Banach space is norm-null iff the BS is finite dimentional in [Josefson: Arkiv for Math (1975)] and [Nissenzweig: Isr. J. Math (1975)]). In particular, B_{c_0} is not limited in c_0.

Phillip's lemma [Phillips: TAMS (1940)] is exactly the fact that $\widehat{B_{c_0}}$ is limited in $c_0'' = \ell^\infty$.

Limited sets are relatively compact in separable and in reflexive Banach spaces by [BD (1984)].

B_{c_0} is DP because $c_0' = \ell^1$ has the Schur property. It is shown by Alpay, Gorokhova, and EE [AEG: preprint (2023)] that DP sets turn to limited sets while embedded in the bi-dual.

Theorem 8. [AEG (2023)] Let $A \subseteq X$. TFAE:

$i)$ A is a DP subset of X.

$ii)$ $\widehat{A}$ is a limited subset of X''.

Dunford–Pettis L-weakly compact and limitedly L-weakly compact operators

The proof of the following theorem is based on Proposition 1.

Theorem 9. [AEG (2023)] *Let $T \in \mathsf{L}(X, F)$. TFAE.*

$i)$ T *takes limited subsets of* X *onto* Lwc *subsets of* F.

$ii)$ T *takes compact subsets of* X *onto* Lwc *subsets of* F.

$iii)$ $\{Tx\}$ *is an* Lwc *subset of* F *for each* $x \in X$.

$iv)$ $T'f_n \stackrel{w^*}{\to} 0$ *in* X' *for each disjoint bounded sequence* (f_n) *in* F'.

Because of i), we prefer to call operators satisfying the above conditions by limitedly Lwc *operators* (they may equally deserve to be called *compactly* Lwc *operators* due to ii)). Operators satisfying iv) were introduced in [Oughajji and Moussa: Afr. Mat. (2022)] under the name *weak* L-*weakly compact operators* (this name looks more suitable for a-Lwc operators rather than for l-Lwc operators).

Definition 10. An operator $T : X \to F$ is called:

a) a *Dunford–Pettis* L-*weakly compact* (briefly, $T \in$ DP-Lwc(X, F)), if T carries DP subsets of X onto Lwc subsets of F.

b) *limitedly* L-*weakly compact* (briefly, $T \in$ l-Lwc(X, F)), if T carries limited subsets of X onto Lwc subsets of F.

DP-Lwc(X, F) and l-Lwc(X, F) are vector spaces. Theorem 9 ii) provides the second inclusion of the next formula, whereas the first one is trivial.

$$\text{Lwc}(X, F) \subseteq \text{a-Lwc}(X, F) \subseteq \text{l-Lwc}(X, F).$$

Following [Emmanuele: Indiana Univ. Math. J. (1987)], a Banach space X is said to possess the *Bourgain–Diestel property* if each limited subset of X is relatively weakly compact, and an operator $T : X \to Y$ is called a *Bourgain–Diestel operator* (briefly, $T \in$ BD(X, Y)) if T carries limited sets onto relatively weakly compact sets. The weakly compactness of Lwc sets, Definitions 7, 10, and Theorem 9 imply

$$\text{Lwc}(X, F) \subseteq \text{DP-Lwc}(X, F) \subseteq \text{l-Lwc}(X, F) \subseteq \text{BD}(X, F).$$

All inclusions here are generally proper.

Example 11. a) $\mathrm{Id}_{\ell^1} \in$ a-Lwc$(\ell^1)\backslash$Lwc(ℓ^1) because relatively weakly compact subsets of ℓ^1 are almost order bounded, and they in turn are Lwc.

b) $\mathrm{Id}_{\ell^2} \in$ l-Lwc$(\ell^2) \setminus$ a-Lwc(ℓ^2) since limited sets in ℓ^2 coincide with relatively compact sets that are in turn l-Lwc sets in ℓ^2, while B_{ℓ^2} is weakly compact but not an l-Lwc set.

c) $T := \mathrm{Id}_{c_0} \in$ l-Lwc(c_0), yet

$$T'' = \mathrm{Id}''_{c_0} = \mathrm{Id}_{\ell^\infty} \notin \text{l-Lwc}(\ell^\infty) = \text{l-Lwc}(c''_0).$$

d) Since l-Lwc$(\ell^2) =$ DP-Lwc(ℓ^2) due to reflexivity of ℓ^2, item b) implies $\mathrm{Id}_{\ell^2} \in$ DP-Lwc$(\ell^2) \setminus$ a-Lwc(ℓ^2). We have no example of an operator $T \in$ a-Lwc$(X, F) \setminus$ DP-Lwc(X, F).

e) $\mathrm{Id}_{c_0} \in$ l-Lwc$(c_0) \setminus$ DP-Lwc(c_0) as B_{c_0} is not Lwc yet is a DP set in c_0.

f) $\mathrm{Id}_c \in$ BD$(c) \setminus$ l-Lwc(c) since limited sets in c coincide with relatively compact sets, while $c^a = c_0 \underset{\neq}{\subsetneq} c$ implies $\mathrm{Id}_c \notin$ l-Lwc(c) by Theorem 9.

g) Combining d)–f) in one diagonal operator (3×3)-matrix:

$$\mathrm{Lwc}(\ell^2 \oplus c_0 \oplus c) \underset{\neq}{\subsetneq} \text{DP-Lwc}(\ell^2 \oplus c_0 \oplus c) \underset{\neq}{\subsetneq} \text{l-Lwc}(\ell^2 \oplus c_0 \oplus c) \underset{\neq}{\subsetneq} \mathrm{BD}(\ell^2 \oplus c_0 \oplus c).$$

The equivalence $i) \iff ii) \iff iii) \iff iv)$ below follows from Theorem 9.

Theorem 12. [AEG (2023)] *Let* $T \in \mathsf{L}(X, F)$. *TFAE:*

$i)$ $T'' \in$ l-Lwc(X'', F'').

$ii)$ T'' *takes compact subsets of* X'' *to* Lwc *subsets of* F''.

$iii)$ $T''(X'') \subseteq (F'')^a$.

$iv)$ $T''' f_n \stackrel{w^*}{\to} 0$ *in* X''' *for each disjoint bounded* (f_n) *in* F'''.

Each of above equivalent conditions implies:

$v)$ $T \in$ DP-Lwc(X, F).

The condition $v)$ *in turn implies:*

$vi)$ $T \in$ l-Lwc(X, F).

Note that, $T \in$ l-Lwc(X, F) does not imply $T'' \in$ l-Lwc(X'', F'') in general (see Example 11 c)). If $T'' \in$ DP-Lwc(X'', F'') then $T'' \in$ l-Lwc(X'', F''), and hence $T \in$ DP-Lwc(X, F) by Theorem 12. We have no example of an operator $T \in$ DP-Lwc(X, F) such that $T'' \notin$ DP-Lwc(X'', F'').

Limitedly M-weakly compact operators and the semi-duality theorem

The following definition was taken a starting point in [Oughajji and Moussa: Afr. Mat. (2022)]. In our approach, this definition is a derivation of Theorem 9 iv), similarly to the classical approach to Mwc operators introduced in [Meyer-Nieberg: Math. Z. (1974)] as a derivation of Lwc operators.

Definition 13. An operator $T : E \to Y$ is *limitedly* Mwc (briefly, $T \in$ l-Mwc(E,Y)), if $Tx_n \stackrel{\text{w}}{\to} 0$ for every disjoint bounded sequence (x_n) in E.

Now, we discuss the semi-duality for l-Lwc and l-Mwc operators. It was proved in [Oughajji and Moussa (2022)] that $T \in$ l-Mwc$_+(E, F)$ iff $T' \in$ l-Lwc$_+(F', E')$. The next theorem give the general case.

Theorem 14. [AEG (2023)] *The following statements hold:*

i) $T' \in$ l-Mwc$(F', X') \Rightarrow T \in$ l-Lwc(X, F).

ii) $T' \in$ l-Lwc$(Y', E') \Leftrightarrow T \in$ l-Mwc(E, Y).

Доказательство. i) Let $T' \in$ l-Mwc(F', X'), and let (f_n) be disjoint bounded in F'. Then $T'f_n \overset{\text{w}}{\to} 0$, and hence $T'f_n \overset{\text{w}^*}{\to} 0$. Theorem 9 implies $T \in$ l-Lwc(X, F).

ii) ($\Leftarrow$): Let $T \in$ l-Mwc(E, Y). By Theorem 9, for $T' \in$ l-Lwc(Y', E'), we need to prove that $\{T'f\}$ is an l-Lwc subset of E' for each $f \in Y'$. Let $f \in Y'$. By Lemma 2, it suffices to show $f(Tx_n) \to 0$ for each disjoint sequence (x_n) in B_E. So, let (x_n) be disjoint in B_E. Since $T \in$ l-Mwc(E, Y) then $Tx_n \overset{\text{w}}{\to} 0$, and hence $f(Tx_n) \to 0$, as desired.

($\Rightarrow$): Let $T' \in$ l-Lwc(Y', E'). Then $\{T'g\}$ is an Lwc subset of E' for each $g \in Y'$ by Theorem 14. It follows from Lemma 2 that $g(Tx_n) = T'g(x_n) \to 0$ for each disjoint bounded sequence (x_n) in E. Since $g \in Y'$ is arbitrary, $Tx_n \overset{\text{w}}{\to} 0$ for every disjoint bounded (x_n) in E, and therefore $T \in$ l-Mwc(E, Y). $\square$

The similar semi-duality was established in Theorem 6 by Bouras, Lhaimer, and Moussa [Positivity (2018), (2021)] for almost L-weakly compact operators and for order L-weakly compact operators.

Although, we have no sequential characterization of DP-Lwc operators like the characterization of l-Lwc operators given by Theorem 9 iv), there is the following result in this direction.

Theorem 15. [AEG (2023)] *Let* $T \in \mathrm{L}(X, F)$. *TFAE.*

i) $T'' \in$ DP-Lwc(X'', F'').

ii) $T' f_n \stackrel{\mathrm{w}}{\to} 0$ *in* X' *for each disjoint* (f_n) *in* $B_{F'}$.

Clearly, DP-Lwc(X, F), l-Lwc(X, F), and l-Mwc(E, Y) are vector spaces. It is natural to ask whether or not DP-Lwc(X, F), l-Lwc(X, F), and l-Mwc(E, Y) are Banach spaces under the operator norm. The answer is affirmative.

Theorem 16. [AEG (2023)]

$i)$ *If* DP-Lwc$(X, F) \ni T_n \overset{\|\cdot\|}{\to} T$ *then* $T \in$ DP-Lwc(X, F).

$ii)$ *If* l-Lwc$(X, F) \ni T_n \overset{\|\cdot\|}{\to} T$ *then* $T \in$ l-Lwc(X, F).

$iii)$ *If* l-Mwc$(E, Y) \ni T_n \overset{\|\cdot\|}{\to} T$ *then* $T \in$ l-Mwc(E, Y)

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